Abstract

In this study a numerical method for general applications with non-Newtonian fluids is developed to investigate the pure squeeze motion in an isothermal elastohydrodynamic lubricated spherical conjunction under constant load conditions. The coupled transient modified Reynolds, the elasticity deformation, and the load equilibrium equations are solved simultaneously. Computer simulation is carried out to investigate the effects of flow rheology and operations on the relationship between the pressure and film thickness distributions. The simulation results reveal that the larger the flow index \( n \), the larger the film thickness and the smaller the maximum central pressure. This results in larger time needed to obtain maximum central pressure. In addition, the elastic deformation is more significant for the lower flow index. Therefore, the smaller the flow index becomes, the greater the difference between the hydrodynamic lubrication (HL) solution and elastohydrodynamic lubrication (EHL) solution becomes.

Keywords: EHL; Squeeze film; Power law lubricants

1. Introduction

When two bodies approach each other along a normal direction, a very high pressure will be generated in the lubricating film due to the squeeze effects, and an elastic dimple will occur at the center of the contact region. The problems related to this phenomenon are called transient elastohydrodynamic lubrication (EHL) problems. These problems occur in many mechanical elements making contact in pairs under high pressure, such as gear teeth, cams and followers, piston rings and cylinder, rolling element bearings, and the stretching process of sheet metal. In addition, when lubricants contain a large quantity of high molecular weight polymers as a viscosity index improver, such as liquefied gas lubricants and bodily fluids like blood and synovia, the non-Newtonian characteristics of lubricants become important.

Christensen [1] numerically studied the pure squeeze film problem lubricated by an incompressible fluid with a viscosity varying exponentially with pressure. Lee and Cheng [2] developed an interesting numerical scheme for pure squeeze EHL problems lubricated by a compressible fluid with a viscosity varying arbitrarily with pressure. In the analysis of a ball dropping problem, Yang and Wen [3,4] solved the equation of the motion of the ball to determine its position during impact. However, the rebound was not studied, and the primary peak was not reached in their analysis. Dowson and Wang [5] analyzed the bouncing of an elastic sphere on an oily plate. Their analysis was restricted to normal motion, in order to develop a numerical procedure, and to relate the overall findings to the results presented by Safa and Gohar [6]. Larsson and Höglund [7,8] showed that the maximum pressure in a lubricant film could reach levels higher than in the corresponding
The effects of flow rheology on the EHL pure squeeze film problem, a non-Newtonian power-law fluid is utilized as the lubricant. A numerical method for general applications is developed to investigate the pure squeeze action in an isothermal EHL spherical conjunction. The transient modified Reynolds equation as well as the elasticity equation are solved numerically. The effects of flow rheology of the lubricant as well as the elastic properties of solids on the performance of the squeeze film are proposed and discussed under constant load condition.

2. Theoretical analysis

2.1. Modified Reynolds equation

Two spheres approaching one another can be expressed as the equivalent sphere approaching a plane. Consider the squeezing film mechanism as shown in Fig. 1, where an elastic sphere of radius \( R \) is approaching in an infinite plate with a velocity under constant load. The lubricant in the system is taken to be a non-Newtonian power-law lubricant.

The equations of motion governing the axial symmetric flow of a compressible fluid under the assumptions of the lubrication theory, and neglecting the inertia terms, in one dimension, are given by

\[
\frac{dp}{dr} = \frac{\hat{c}_r}{\hat{c}_z}.
\]

Fig. 1. Geometry of EHL of circular contacts under pure squeeze motion.

<table>
<thead>
<tr>
<th>Nomenclature</th>
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<td>( \bar{\rho} )</td>
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\[
\frac{dp}{dz} = 0. \tag{2}
\]

Since the flow occurs under the pressure gradient only and the pressure is constant across the film. Mathematically, the constitutive equation of power-law fluid under the present axial symmetric problem is expressed as [15]

\[
\tau = m \left[ \frac{\partial v_r}{\partial z} \right]^{n-1} \frac{\partial v_r}{\partial z}, \tag{3}
\]

where \( \tau \) is the shear stress, \( \partial v_r/\partial z \) the shear rate, \( m \) the consistency index, and \( n \) is the flow index. The conditions of \( n > 1 \), \( n = 1 \), and \( n < 1 \) correspond to a dilatant fluid, Newtonian fluid, and pseudoplastic fluid, respectively.

Substituting the value of \( \tau \) from Eq. (3) into Eq. (1), the equation of motion becomes

\[
\frac{dp}{dr} = \frac{\partial}{\partial z} \left[ m \left[ \frac{\partial v_r}{\partial z} \right]^{n-1} \frac{\partial v_r}{\partial z} \right]. \tag{4}
\]

The boundary conditions are

\[
v_r(t, r, 0) = v_r(t, r, h) = 0.0, \quad \frac{\partial v_r}{\partial z}(t, r, h/2) = 0.0, \tag{5}
\]

\[
v_z(t, r, 0) = 0.0, \quad \frac{\partial v_z}{\partial t}(t, r, h) = \frac{\partial h}{\partial t}. \tag{6}
\]

Integrating Eq. (4) with the boundary conditions (5), the velocity is obtained as

\[
v_r = m^{-1/(n-1)} \left( n + 1 \right) \left[ \frac{dp}{dr} \right] \left[ \frac{\partial v_r}{\partial z} \right]^{(1-n)/n} \left( h_z \right) \left( n+1 \right) \left[ \frac{\partial h}{\partial z} \right] \left( 1+(n+1)/n \right). \tag{7}
\]

The equation of continuity is

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \rho v_r \right) + \frac{\partial}{\partial z} \left( \rho v_z \right) = 0. \tag{8}
\]

Substituting the value of \( v_r \) from Eq. (7) into Eq. (8) and integrating across the film, we have

\[
\frac{\partial}{\partial r} \left[ \rho h^{(2n+1)/n} \left( - \frac{1}{m} \frac{dp}{dr} \right)^{1/n} \right] = -\zeta \frac{\partial h}{\partial t}, \tag{9}
\]

where

\[
\zeta = \frac{2n+1}{2n}, \tag{10}
\]

or in dimensionless form as

\[
\frac{\partial}{\partial X} \left[ \bar{\rho} X h^{(2n+1)/n} \left( - \frac{1}{m} \frac{\partial \bar{P}}{\partial X} \right)^{1/n} \right] = -\xi \frac{\partial \bar{H}}{\partial t}, \tag{11}
\]

where

\[
\xi = \frac{\zeta (1/n)}{(1.5 W_0)^{(n+2)/3n}}. \tag{12}
\]

The radial coordinate, \( X \), has its origin at the center of the contact. The boundary conditions for Eq. (11) are

\[
P(X \to \infty, T) = 0, \tag{13a}
\]

\[
\frac{\partial}{\partial X} P(0, T) = 0, \tag{13b}
\]

\[
P(X, T) \geq 0. \tag{13c}
\]

2.2. Hydrodynamic lubrication

During the squeeze motion, two stages are included, i.e. the initial stage and the high-pressure stage. At the initial stage, the ball has achieved the lubricant layer and begins to squeeze the lubricant film away. Since the pressure is low, an isoviscous incompressible lubricant model is appropriate and the elastic deformation can be disregarded. In this stage, the transient modified Reynolds equation in dimensionless form can be expressed as:

\[
\frac{\partial}{\partial X} \left[ X \bar{h}^{(2n+1)/n} \left( - \frac{\partial \bar{P}}{\partial X} \right)^{1/n} \right] = -\xi X \frac{\partial \bar{H}}{\partial T}. \tag{14}
\]

The film thickness can be expressed as:

\[
\bar{h} = H_0 + \frac{X^2}{2}, \tag{15}
\]

and the central normal velocity is simply given as

\[
\frac{\partial \bar{H}}{\partial T} = \frac{\partial H_0}{\partial T} = V_{z0}. \tag{16}
\]

At the initial stage, the equation governing the transient hydrodynamic lubrication (HL) problem is numerically solved when a thin layer of oil initially separates a ball and a plate with a pure squeeze motion.

2.3. Elastohydrodynamic lubrication

When the pressure increases with time, the elastic deformation, and the effect of pressure on the viscosity cannot be neglected. This stage is denoted as the high-pressure stage. It is referred to as the problem of pure squeeze motion in EHL. The coupled Reynolds, rheology, and elasticity equations have to be solved numerically.

Since both the viscosity and density of the lubricant are assumed to be functions of pressure only, the relationship between viscosity and pressure used by Roelands [16] can be expressed as:

\[
\frac{\phi}{\rho_0} = 1 + \frac{0.6 \times 10^{-9} p}{1 + 1.7 \times 10^{-9} p}. \tag{18}
\]

where \( \phi \) is the viscosity at the ambient pressure and \( \zeta \) is the pressure–viscosity index. According to Dowson and Higginson [17], the relationship between density and pressure is given as:

\[
\frac{\rho}{\rho_0} = \exp \left[ (9.67 + \ln n_0) \left[ -1 + (1 + 5.1 \times 10^{-9} p) \right] \right], \tag{17}
\]

where \( n_0 \) is the viscosity at the ambient pressure and \( p \) is the pressure–viscosity index.
The film thickness in a nominal point contact elastohydrodynamic conjunction can be written as

\[ h(r, t) = h_0(t) + \frac{r^2}{2R} + \delta(r, t). \]  

(19)

To calculate the static deformation due to pressure distribution, influence coefficients \( D_{ij} \) are introduced. The deformation can thus be computed at discrete points \( i \) as a sum of the deformation contributions from all pressure points \( j \):

\[ \delta_i = \sum_{j=1}^{k} D_{ij} P_j. \]  

(20)

The dimensionless film thickness between two elastic bodies in circular contacts can be expressed as

\[ H_i = H_0 + \frac{X_i^2}{2} + \sum_{j=1}^{k} D_{ij} P_j, \]  

(21)

where the influence coefficients, \( D_{ij} \), are computed according to Yang and Wen [5], and Larsson and Högland [10].

The instantaneous load balance equation for a constant load condition is

\[ \int_0^\infty PX \, dX = \frac{1}{3}. \]  

(22)

The rigid separation is an unknown variable in each time step. It can be determined by solving the transient Reynolds equation with the load balance equation.

### 3. Results and discussion

In this section, the ball is assumed to accelerate continuously from the free surface of the lubricant layer \( (h_0 = 20 \mu m) \) until it achieves a quasi-static condition. To discuss the effects of flow rheology and elastic deformation on the squeezing motion, the point contact EHL problems are discussed under the conditions of non-isoviscous, compressible lubricant, and constant load. Numerical solutions of film profile and pressure distribution in pure squeeze motion are calculated using the parameters listed in Table 1.

For the present constant load case, the rigid separation cannot be determined by the equation of motion. Hence, the load equilibrium equation has to be included in the coupled transient modified Reynolds equation, Eq. (11), and the elastic deformation equation, Eq. (21). The rigid separation becomes one of the unknown variables, and will be solved simultaneously with the nodal pressures. The upper limit of the computational region in the beginning is chosen as \( X_{max} = 16.0 \). When more than half of the region is cavitated, the maximum analyzed region \( X_{max} \) reduces to half of its initial region, and so on, until \( X_{max} = 2.0 \). The grid is made up of 401 nodes. The Gauss–Seidel iteration is employed to calculate the film thickness and pressure distribution at each time step.

### Table 1

<table>
<thead>
<tr>
<th>Computational data</th>
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<tbody>
<tr>
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<tr>
<td>Properties of lubricants</td>
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<tr>
<td>Lubricants type</td>
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<td>G (Material parameter)</td>
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<td>Inlet viscosity of lubricant (Pa-s)</td>
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<td>Inlet density of lubricant (kg/m³)</td>
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<td>Pressure–viscosity coefficient (1/GPa)</td>
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<td>Pressure–viscosity coefficient (Roelands)</td>
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<td>Properties of balls</td>
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<td>Density of balls (kg/m³)</td>
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<td>Elastic modulus of balls (GPa)</td>
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<td>Poisson's ratio of balls</td>
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Under the conditions of Newtonian fluid \( (n = 1.0) \) and constant load, the operation and initial conditions of Yang and Wen [4] are employed in the present algorithm to solve the pure squeeze EHL motion of the circular contacts problem. The proposed algorithm’s numerical results of central pressure and film profile are compared with those obtained by Yang and Wen [4] as shown in Fig. 2, and indicate a good agreement. The discrepancies come from the finer grids and calculation region varying with time in the present analysis.

As shown in Fig. 3, central pressure \( (p_c) \) and central film thickness \( (h_c) \) are plotted as functions of time for various flow indices \( (n = 0.9, 1.0, 1.1) \). As squeeze starts, the central pressure and the central film thickness obtained by the HL model are slightly smaller than those from the EHL model. As the ball approaches the plate closer, we have an increasing \( p_c \) and a decreasing \( h_c \) and the elastic deformation effects become significant. The HL and EHL models are also compared for various flow indices as shown in Table 2. In the initial stage, when the discrepancy of \( h_c \) estimated by the HL and EHL models, approach 5%, the discrepancy of \( p_c \) between the two models is smaller than 2.5%. When the discrepancy of \( p_c \) estimated by the HL and EHL models is almost the same, the discrepancy of \( h_c \) between the two models increases to about 30%. The difference between the HL solution and the EHL solution using pseudoplastic fluid \( (n = 0.9) \) is within 5% when the central film thickness is greater than 3.25 µm. The difference between the HL solution and the EHL solution using dilatant fluid \( (n = 1.1) \) is within 5% when the central film thickness is greater than 2.68 µm. It is evident from Fig. 3 that the elastic deformation is more significant for lower flow index. Therefore, the smaller the flow index, the greater the difference between the HL solution and EHL solution.

In the case with constant load condition, Figs. 4 and 5 show the relative change in the dimensionless pressure distribution and dimensionless film thickness for a flexible sphere approaching a lubricated flat surface for three different lubricants where \( \mathcal{W} = 5.24 \times 10^{-8} \) and \( G = 3500 \).
It can be seen from Fig. 4 that the pressure profile of the Newtonian fluid ($n = 1.0$) is quite flat at a relatively large film thickness, but it becomes steeper as the film thickness decreases. When the sphere approaches the flat surface, the pressure profile is almost converged to the well-known Hertzian contact pressure. As shown, the peak pressure is always kept at the center in this study. It is found that the center pressure increases gradually with the decreasing central film thickness from 0.00023 to 0.00135 s as the central dimensionless film thickness decreases to a certain level (between 0.4 and 0.5). After this stage, the pressure reverses its trend from 0.00135 to 0.38 s, i.e., the peak pressure decreases with the decreasing film thickness until it reaches a stage where the minimum film thickness and the squeeze velocity are almost zero. In this stage, the pressure gradient is significantly influenced by the entraining motion term where the squeeze velocity becomes smaller. Since the squeeze velocity at 0.38 s is almost zero, it is interesting to note that the peak pressure and the central film thickness at 0.38 s are smaller than those at 0.00135 s due to almost no...
squeeze motion effect. From Fig. 5, the position of minimum film thickness departs further from the center \((r = 0)\).

To discuss the effects of flow rheology and elastic deformation on the squeezing motions, three flow indices are considered, i.e. \(n = 0.9, 1.0, \) and \(1.1\) respectively. The dimensionless pressure distributions and the dimensionless film thickness are plotted as time varying for the three different fluids as shown in Figs. 4 and 5, respectively. At the initial squeeze stage, smaller flow index \((n)\) results in larger central pressure as compared to that of larger flow index. At the same time, the region outside of the central region will reveal contrary results. These phenomena come from the conditions of constant load. In addition, the film thickness distribution decreases as the flow index decreases. At \(3.9 \times 10^{-3}\) s, the maximum central pressure for the pseudoplastic fluid \((n = 0.9)\) is reached first, and the dimple effects are significant compared to the other two lubricants \((n = 1, \) and \(n = 1.1)\). As time increases, the pressure spikes and dimples of the pseudoplastic fluid retard. Beyond \(1.35 \times 10^{-3}\) s, the larger the flow index, the larger the central pressure. Similarly, the region outside of the central region will reveal contrary results. As time goes on, the central pressure of the dilatant fluid \((n = 1.1)\) reaches its maximum, and then reaches the Hertz contact condition. The most significant dimple effect occurs at the time of maximum pressure.

Fig. 6 shows the pressure and the film thickness at the contact center versus time for three different lubricants under constant load condition. At the initial stage the central pressure increases rapidly with time to a maximum, and then decreases slowly to near the amplitude of the well-known Hertzian pressure at the final stage. This stage can be considered as the quasi-static condition. For larger flow index, the central film thicknesses are larger than those for smaller flow index. At the initial squeeze stage, smaller flow index \((n)\) results in larger central pressure compared to that of larger flow index. The time needed to achieve maximum central pressure increases as the flow index \((n)\) increases. The greater the flow index \((n)\) is, the greater the minimum film thickness is.

Fig. 7 shows the maximum pressure and the minimum film thickness at the contact center versus the flow index \((n)\) for \(G = 3500,\) and for the different loads (which are obtained from \(t = 0\) to \(t = 320\) ms). The effects of flow rheology produce an increase in the minimum central film thickness and a decrease in the maximum central pressure as the value of the flow index \((n)\) increases. The larger the load is, the greater the maximum central pressure.
and minimum central film thickness are. For larger loads, the maximum central pressure decreases relatively rapidly with increasing the flow index \( n \) and the minimum central film thickness increases relatively rapidly with increasing the flow index \( n \) as compared to those with lower loads.

In Fig. 8, the central pressure is plotted as a function of the central film thickness for various flow indices, load conditions, and material parameters. The central pressure increases as the central film thickness decreases. As the sphere approaches closer to the plate, the central pressure increases abruptly, increasing to the peak value, then decreasing abruptly from the peak, and finally decreases down to the Hertz pressure. The results show that the slope value of \( p_c \) versus \( h_c \) near the peak is greater for larger material parameters, or larger load, or smaller flow index.

Fig. 9 shows the relationship of the central normal squeeze velocity \(-V_c\) and the central film thickness at a specified flow index \( n \), load, and materials parameter. At the initial impact stage, the central normal squeeze velocity decreases rapidly with decreasing central film thickness at higher loads and low flow index \( n \). At the high-pressure stage, the central normal squeeze velocity decreases rapidly with decreasing central film thickness between 0.1 and
0.6 μm due to the elastic deformation at higher loads and low flow index \( (n) \). Meanwhile, the central pressure increases rapidly to a maximum. Then the central pressure decreases quickly to near the amplitude of the well-known Hertzian pressure with decreasing central film thickness. In the final stage, the central normal squeeze velocity is less than 10 μm/s and the central pressure gradually decreases to the Hertzian pressure. This stage can be considered a quasi-static condition. It can also happen that the central normal squeeze velocity decreases relatively quickly with decreasing central film thickness in the case of: lower load (case \( B \rightarrow \) case \( A \)), lower flow index (case \( C \rightarrow \) case \( A \)) or lower materials parameter (case \( A \rightarrow \) case \( D \)).

4. Conclusions

A numerical method for general applications with non-Newtonian fluid was developed to investigate the pure squeeze action in an isothermal EHL spherical conjunction under constant load condition. The coupled transient modified Reynolds, elasticity deformation, and the load equilibrium equations were solved simultaneously by using the finite difference method and the Gauss–Seidel iteration method. The effects of power-law lubricants and elastic deformation on the performance of squeeze film were proposed and discussed. Some of the results are summarized as follows:

1. The elastic deformation is more significant for the lower flow index. Therefore, the smaller the flow index becomes, the greater the difference between the HL solution and EHL solution is.
2. The larger the flow index, the larger the film thickness, the smaller the maximum central pressure, and results in larger amount of time needed to reach maximum central pressure.
3. The slope value of the central pressure versus the central film thickness near the peak pressure is greater for larger loads, or larger material parameters, or smaller flow index.
4. In the initial stage, the central normal velocity decreases rapidly with decreasing central film thickness at higher loads and low flow index \( (n) \). In the final stage, for lower flow index, lower load or lower materials parameter, the central normal squeeze velocity decreases relatively quickly with decreasing central film thickness.

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References